A STUDY ON THE APPLICATION OF MONTE CARLO METHOD TO STOCHASTIC OPTIMIZATION

Alexandru Hampu, Assoc. Prof., PhD, Romanian-German University of Sibiu

Abstract.Uncertainty is present in most situations of decision making. Stochastic programming is a useful tool for solving decision problems involving uncertainty. The Monte Carlo method is one of the most popular techniques for generating random numbers. The purpose of this paper is to analyze ways of using the Monte Carlo method for solving stochastic programming problems.

Key word: ***Stochastic programming***

1. Introduction

I will briefly present the history of the appearance and evolution of the Monte Carlo method.This method was attributed in its modern form to Stanislaw Ulam,created while he was working at Los Alamos National Laboratory, New Mexico in the 40s.In the period 1939-1946 the Manhattan Project was conducted in order to produce the nuclear weapon,the research team beeing led by the American physicist J. Robert Oppenheimer. One of the three existing research centers and production project called Los Alamos National Laboratory,situated in Los Alamos, New Mexico,was the place where starting with 1943 a team of scientists was created who worked on the Manhattan Project: Nick Metropolis, John von Neumann, Stnislaw Ulam, Edward Teller, Robert Richtmyer and others.The researches, which were conducted to create the first atomic bomb, led in parallel to scientifical results for various other fields such as physics, mathematics, electrical engineering, computers, chemistry, etc. This is the way this method known as Monte Carlo was born (named as such by Nick Metropolis from roulette casinos which generated random numbers) and attributed in this current form to Stanislaw (Stan) Ulam.

What is the Monte Carlo, which is the aim for which it was created? A brief and clear answer is given by physicist Anderson H. L. [1], one of those who participated in the Manhattan Project and was close to these achievements: " The Monte Carlo method is an application of the laws of probability and statistics to the natural sciences. The essence of the method is to use various distributions of random numbers, each distribution reflecting a particular process in a sequence of processes such as the diffusion of neutrons in various materials, to calculate samples that approximate the real diffusion history. Statistical sampling had been known for some time, but without computers the process of making the calculations was so laborious that the method was seldom used unless the need was compelling".

Historically speaking, the processes underlying the Monte Carlo method had been used since the '30s by Enrico Fermi, an Italian physicist, Nobel laureate for Physics in 1938, the discoverer of nuclear fission. Fermi, who in 1924 published a paper on statistical physics of the particles, used after 1933 statistical sampling in the study of neutron diffusion at the University of Rome, spurred by the discoveries of Frederic Joliot and Irene Curie in the field of radioactivity.One of the main obstacles to the use of these methods was linked to the large volume of calculations that had to be performed, because the calculating machines existing at that time were nonperforming.Fermi still used such a machine manually operated, created by the German company Brunsviga, which he took with him to Columbia University in 1939 when he emigrated to the US and where later he was employed in the team that created the atomic bomb.

What was Stan Ulam's contribution to the finalization of the Monte Carlo method and which factors favored carrying out the method? A first favorable factor was linked to the creation of the first electronic computer ENIAC (Electronic Numerical Integrator And Computer) at the University of Pennsylvania in Philadelphia between 1943-1946. The connection between the created computer to the Los Alamos Laboratory, where Stan Ulam worked, was achieved by John von Neumann. Ulam and Nick Metropolis were invited to try on ENIAC, occasion on which Ulam proposed to Neumann to resuscitate the Monte Carlo method, predicting the possibility of using it in the field of neutronic diffusion, mathematical physics and other disciplines. This idea was accepted by Neumann, who intuited the potential of this method from the viewpoint of a brilliant mathematician, and thus the two, together with Metropolis and other scientists from the group Los Alamos, starting with 1946, developed a series of algorithms derived from statistical sampling.

What essentially performs the Monte Carlo method? It has changed the approach to solving.Before the Monte Carlo method was developed, simulations tested a previously understood deterministic problem and statistical sampling was used to estimate uncertainties in thesimulations. Monte Carlo simulations invert this approach, solving deterministic problems using a probabilistic analog.

There are many ways for using the MC method, of which we mention only two:

- generating values of a random variable by using a pseudorandom number generator with uniform distribution, used in the statistical selection
- determining solutions for integro-differential equations by generating random numbers selection experiments

Since 1949 symposiums on MC method have been organized, first in Los Angeles, then in various other universities in order to find new applications of this extremely useful method. We will briefly enumerate some of the areas of applicability of the MC method:

- Physics (process simulation with many degrees of freedom with random behavior)
- Mathematics (optimization, numerical integration, probabilities)
- Economy and Industry (random process simulation, forecasting degree of risk)
- Finance
- Statistics

and the list can continue.

In this paper we are interested in using the MC method in stochastic optimization, issue that has been aproached by other authors for certain models. The study of this issue remains open, considering that stochastic optimization has rapidly evolved in recent years favored on one hand by the growth of computers and on the other hand by the theoretical accumulation concretized in new models.

2. Application of the Monte Carlo method in stochastic optimization

We consider one of the classical models of stochastic programming i.e. the stochastic programming model with simple recourse (in two stages) formulated by Dantzig and Madansky in 1961; considering on x and y as the vectors of decision for the first and the second stage the decision-observation scheme is the following:

decision on x
observation of ω
decision on y

The solving of the stochastic programming model with simple recourse means solving of the two so called stages and, in order to be easier to note, we begin with the second stage, which is:

$$\min q^{T}y \tag{2.1}$$

subject to:

$$Tx + Wy = \omega$$

$$y \geq 0,$$

where y and q are m_2 – dimensional vectors, T and W are m_2 x n respectively m_2 x m_2 –matrix, $\tilde{\omega}$ is a random vector m_2 –dimensional defined on a field of probability (Ω, K, P) with P probability distribution.

The *first stage* of this problem is:

$$\min \{c^{T}x + Q(x)\} \tag{2.2}$$

subject to:

$$Ax = b$$

$$x \geq 0$$

$$x \in X$$

unde $c = (c_1, c_2, \ldots, c_n)^{T}$, x is an unknown n-dimensional vector, A is an m_1 x n matrix, b ia an m_1- vector, $X=\{x \in \mathrm{R}^{n} |\ Tx+Wy = \omega \text{ a.s}\}$, $Q(x) = \mathbf{E}_{P}\{Q(x, \tilde{\omega})\} = \sum_j p^{j} Q(x, \omega^{j})$, *the recourse function* being $Q(x, \omega) = \min_{y \in Y_{\xi}(x)} q^{T}y$ where $Y_{\xi}(x) = \{y \in \mathrm{R}^{m_2} |\ Tx+Wy = \omega,\ y \geq 0\}$.

We consider $\omega^1, \omega^2, \ldots, \omega^K$, the values for ω taken with probability $p_1, p_2, \ldots, p_K$, that is, we assume that P has a finite support, K being the number of scenarios.

The following mathematical model represents the *nested formulation* of a *multistage linear stochastic program*

$$\min c^1x^1 +$$

$$\mathbf{E}_{\omega^1}\left[\min_{y^1} q^1 y^1(\omega^1) + \mathbf{E}_{\omega^2}\left[\min_{y^2} q^2 y^2(\omega^2) + \ldots + \mathbf{E}_{\omega^{H-1}}\left[\min_{y^H} q^H y^H(\omega^H)\right]\right]\right]$$

subject to:

$$Ax^1 = b$$

$$T^1(\omega^1)x^1 + W^1(\omega^1)y^1 = h^1(\omega^1)$$

$$\vdots$$

$$T^H(\omega^1)x^H + W^H(\omega^1)y^H = h^H(\omega^H)$$

$x^1 \geq 0$, $y^t(\omega^t) \geq$, $t = 1, \ldots, H$

The Monte Carlo method based on sampling has been used in various models of stochastic programming.

There are various techniques of application of the MC method for stochastic programming problem, one being represented by the called interior sample.A. Shapiro [6] shows that this type of sampling is carried within an algorithm, through independent samples generated in the iteration process, recalling one method who can use this technique, stochastic decomposition (Higle and Sen, [2]) .In [2] the method, which is used in step 2 in stochastic decomposition algorithm (SD) for generating independent observations ω^k , $k=1, \ldots, K$. is not specified. In this step of the algorithm the MC method can be used as an alternative.

In general, for a stochastic programming problem with recourse on the type (2.2), the MC method can be applied by generating a random sample $\omega^1, \omega^2, \ldots, \omega^N$ to estimate function $Q(x)$. The need arises because the number of K scenarios can be very high and calculations are extremely difficult. For example, if the random vector ω has 10 independent random components, each with two realizations, then the number of scenarios is 2^{10} , therefore a large number of scenarios. In this case, the function that will be approximated on $Q(x)$ will be of the type

$$\hat{Q}_N(x) = \frac{\sum_{j=1}^{N} Q(x, \omega^j)}{N}$$

and hence we can consider a reasonable way to approximate the objective function of problem (2.2).

Kenyon, A. , Morton, D. P [5] have studied stochastic vehicle routing problem (SVRP). We consider a stochastic vehicle routing problem (SVRP) that consists of planning optimal vehicle routes to service a number of locations in the presence of random travel and service times.

The authors presented the use of the MC method where the scenarios number is too high and deterministic equivalent of SVRP can not be solved exactly.

There are other models where the use of the sample-based MC method is very useful, if not the only solution.

Further, we will present the using of the method MC on a model introduces by the author in [4], named E-model of the multiobjective two-stage stochastic programming problems with simple recourse which is a class 2.1-2.2 models.

$$\min \quad z(x) + \mathbf{E}[Q(x,\omega)] \cdot \mathbf{1} \qquad (2.3)$$

subject to:

$P(Ax \leq \xi_1(\omega)) \geq p_1$

$P(Tx + Wy \leq \xi_2(\omega)) \geq p_2$

$x \geq 0$, $y \geq 0$,

where $z(x) = (z_1(x), z_2(x), \ldots, z_r(x))^T$, $z_i(x) = c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n$, $1 \leq i \leq r$, is linear function, $\mathbf{1} = (1,1,\ldots,1)^T$ is a r-vector, $\xi_1 = (\xi_{11}, \xi_{12}, \ldots, \xi_{1m_1})^T$, $\xi_2 = (\xi_{21}, \xi_{22}, \ldots, \xi_{2m_2})^T$ are random vectors defined on the probability space (Ω,$\boldsymbol{K}$,P), $\omega \in \Omega$, A is an m_1 x n matrix, T is an m_2x n matrix, W is an m_2x m_2 matrix, x is an n-vector, y is an m_2-vector, **E** denotes the mean operator, p_i, i=1,2 is a prescribed probability, $Q(x,\omega)$= min $\{q^T y : P(Tx+Wy \leq \xi_2(\omega))\geq p_2, y\geq 0\}$, q is an m_2-vector. The matrices A and T might have random elements too.

In such a model, there arises the problem of determinationof the deterministic equivalent, both for objective function and to constraints of problem. In the cited article methods for determining the deterministic equivalent were presented. There was no analysis of the situation in which the number of scenarios, however, is too large considering the form of the objective function and restrictions of the problem, in which case it is difficult to determine and analyze the efficient solution. We believe that in such a situation the MC method is applicable.

As I stated above, for the objective function MC sampling can be applied by generating a random sample ω^1, ω^2, …,ω^N to estimate function $Q(x, \omega)$ and allowing the approximation of objective function. For constraints of the problem, a solution could be the use of suitably modified MCSVRP algorithm presented in [5], which allows to estimate the solutions of the variables in an acceptable confidence interval.

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