On the Postulates for Lattices

Graţiela Laslo

Abstract

This paper is inspired from a seria of papers written by J.A. Kalman, in 1955-1959, having the subject the postulates for lattices. It refers especially to systems formed from absorption, idempotence and associativity laws. Following the Kalman's indications from [1], we studied an extended system of axioms, finding all the implications between the subsets of this system.

Introduction

Let us denote by $\mathcal{L}$ the family of all algebraic systems $l = (L, \wedge, \vee)$ consisting of a set L, together with two binary operations on it and let ω be the following set of axioms :

(1)	$x \wedge (x \vee y) = x$	(5)	$x \vee (x \wedge y) = x$
(2)	$x \vee (y \wedge x) = x$	(6)	$x \wedge (y \vee x) = x$
(3)	$(y \vee x) \wedge x$	(7)	$(y \wedge x) \vee x = x$
(4)	$(x \wedge y) \vee x = x$	(8)	$(x \vee y) \wedge x = x$
(A)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(B)	$x \vee (y \vee z) = (x \vee y) \vee z$
(C)	$x \wedge y = y \wedge x$	(D)	$x \vee y = y \vee x$
(I)	$x \wedge x = x$	(J)	$x \vee x = x$

and for each $\xi \in \omega$, let $\mathcal{L}_\xi$ be the family of all l in $\mathcal{L}$ such that l obeys all the laws in ξ. Sorkin considered the set ω in [5] §2, and found all the subsets of ω which constitute an independent set of axioms for lattices. For each $\xi \subseteq \omega$, $\mathcal{L}_\xi$, is a family of generalized lattices. Mostly in the years'60 and '70, a few mathematicians studied noncommutative generalizations of lattices: S.I. Matsushuita, P. Jordan, M.D. Gerhardts, H. Alfonz and nowadays J. Leech, R.J.Bignall, Gh. Fărcaş, Matthew Spinks, Karin Cvetko-Vah, From these, Gh. Fărcaş and J. Leech have helped permanently author of this paper in studying these structures.

About the postulates for lattices have also been written books we mention here "Axiomele laticilor şi algebrelor booleene" ("The Axioms of Lattices and Boolean Algebras") by S. Rudeanu.

In studying noncommutative generalizations for lattices, there were considered different systems of axioms included in. The study of the system ω made by Kalman in [1] is useful even today, as much as the study of an enlarged system denoted by ω_+, having in addition two axioms (A_0) and (B_0), which are weaker than the associativity of " $\wedge$ " and " $\vee$ ".

Kalman mentioned that, at the beginning of the study of any family $\mathcal{L}_\xi$, the following problems arise: P_ξ to find all the subsets of ω that constitute an independent system of axioms for the family $\mathcal{L}_\xi$ and Q_ξ to find all the laws (X) in ω which are obeyed by every l in $\mathcal{L}_\xi$. Sorkin has solved the problem P_ω and the problem Q_ω is trivial since $\mathcal{L}_\omega$ coincide all the class of lattices. Kalman proved results that essesntially solve all the problems P_ξ and Q_ξ , with $\xi \subseteq \omega$. He gave a table presenting "what results" from each independent subset of ω. The table has 95 lines and contains 351 such implications.

In [2] and [3] we studied problems conected with the problems P_ξ and Q_ξ with $\xi \subseteq \omega$: the idempotency in systems of the form $\mathcal{L}[\_\ \_\ ]$, namely the systems $\mathcal{L}_\xi$ where ξ is constituted from two absorption laws, the problem of commutativity of $\wedge$ and $\vee$ in the systems of the form $\mathcal{L}[\_\ \_\ \_\ \_\ ]$, the relations among the systems of the form $\mathcal{L}[\_\ \_\ \_\ \_\ ]$. This study was done using direct proofs and counterexamples.

Also Kalman considered weaker axioms than associativity :

$$(A_0)\ x \wedge (y \wedge x) = (x \wedge y) \wedge x \quad \text{and} \quad (B_0)\ x \vee (y \vee x) = (x \vee y) \vee x$$

If we add to the system ω these two axioms, then the study of $\omega_+ = \omega \cup \{(A_0), (B_0)\}$ is again interesting. For instance we are interested to find out what absorption, idempotency, commutativity axioms result from a certain system of absorption, idempotency, commutativity axioms. We want to see when the associativity axioms are essential in obtaining a certain result and when can they be replace by weaker axioms. The study of this extended system of axioms was proposed by Kalman in [1] §3 and is made by me in the present paper.

In the first section we make some remarks on the results obtained in [2] and [3] and the results obtained by Kalman concerning the subsets of absorption identities. The second section (Closure operations) presents some elementary results concerning closure operations on a given complete lattice. In this second section we define also the closure operator a, which will be essential for solving the proposed problem.

The third section presents the compatibility of a with a group of automorphisms. The fourth section contains the main results of the paper. The preparing results which are the four lemmas were given by Kalman in [1] and proved here by me. These helped me to establish Theorem 1 and Theorem 2 for ω_+ and make the program for proving them.

1 The seven classes of noncommutative lattices

As we said in the previous section, in [2] were presented some classes of algebraic structures generalizing lattices, of the form $\mathcal{L}[\_\,\_]$, namely $\mathcal{L}_\xi$ where ξ is constituted from four absorption laws. More precisely were considered groups of four absorptions, two having the operation $\wedge$ outside the brackets and the other two, the operation $\vee$ outside the brackets. For instance

$$a \wedge (a \vee b) = a, \quad (a \vee b) \wedge a = a, \quad a \vee (a \wedge b) = a, \quad (b \wedge a) \vee a = a.$$

We attempted to answer how many groups of such absorption laws we have, namely how many essentially different classes of noncommutative generalizations of lattices define they. We considered the following relation between two systems of the described type:

$$S_1 \sim S_2 \Leftrightarrow S_1 = S_2 \quad \text{or} \quad (S_1)^{\wedge} = S_2 \quad \text{or} \quad (S_1)^{\vee} = S_2 \quad \text{or} \quad (S_1)^{\wedge\vee} = S_2.$$

that is: S_1 is equivalent with S_2 iff S_1 is equal to S_2 or S_2 can be obtained from S_1 by interchanging the performing order of " $\wedge$ ", or " $\vee$ ", or of both operations. I found twelve equivalence classes having the representatives: $S'_{\substack{\wedge\\ \wedge}}, S_{\substack{\wedge\\ \vee}}, S_{\substack{\wedge\\ \wedge\vee}}, S_{\substack{\vee\\ \wedge}}, S_{\substack{\vee\\ \vee}}, S_{\substack{\vee\\ \wedge\vee}}, S_{\substack{\wedge\vee\\ \wedge\vee}}, S'_{\substack{\wedge\vee\\ \wedge\vee}}, S_{\substack{\wedge\\ \wedge}}, S_{\substack{\wedge\vee\\ \vee}}, S'_{\substack{\vee\\ \vee}}, S_{\substack{\wedge\vee\\ \wedge}}$ (see tabel 1).

From these $S'_{\substack{\wedge\vee\\ \wedge\vee}}$, is the strongest because the algebraic structures defined by $S'_{\substack{\wedge\vee\\ \wedge\vee}}$ has both operations commutative, $S'_{\substack{\wedge\\ \wedge}}$ and $S'_{\substack{\vee\\ \vee}}$ have just one operation commutative. From these twelve, the last four are the dual of other four: $S_{\substack{\wedge\\ \wedge}} = (S_{\substack{\vee\\ \vee}})^*$, $S_{\substack{\wedge\vee\\ \vee}} = (S_{\substack{\wedge\\ \wedge\vee}})^*$, $S_{\substack{\wedge\vee\\ \wedge}} = (S_{\substack{\vee\\ \wedge\vee}})^*$, $S'_{\substack{\vee\\ \vee}} = (S_{\substack{\wedge\\ \wedge}})^*$. (* means the dual of). Thus we have in fact seven essentially different classes of noncommutative lattices.

Kalman in his paper [1] considered different the equivalence relation between the systems of axioms:

$S_1 \sim S_2 \Leftrightarrow \exists\, \sigma \in P$ such that S_2 is the image of S_1 by σ.

Here P is a group of permutations generated by two certain permutation of the axioms from ω. In fact, the Kalman's definition can be described thus:

$$\begin{aligned} S_1 \sim S_2 \quad \Leftrightarrow \quad & S_1 = S_2 \quad \text{or} \quad (S_1)^{\wedge} = S_2 \quad \text{or} \quad (S_2)^{\wedge} \quad \text{or} \quad (S_1)^{\wedge\vee} = S_2 \quad \text{or} \\ & S_1^* = S_2 \quad \text{or} \quad (S_1^*)^{\wedge} = S_2 \quad \text{or} \quad (S_2^*)^{\wedge} \quad \text{or} \quad (S_1^*)^{\wedge\vee} = S_2 \end{aligned}$$

For instance, if we want to see the equivalence class of the system formed from the axioms (1) and (2), system denoted by [12], we know that the axiom (1) can become any other absorption axiom, and also (2) by the following sketch:

Thus, the equivalence class of [12] is $[12] = \{[12], [85], [34], [67], [56], [41], [78], [23]\}$.

If we examine the Kalman's table 2 from [1], he obtained 12 equivalence classes having representatives with four absorption laws. We present below an extract from this table, containing them:

1.	1234	$\overline{S'_{\wedge\vee}}$		
2.	1235	$\underset{\wedge\vee}{\overset{\wedge\vee}{\overline{S_{\wedge\vee}}}}$	and	$\underset{\wedge\vee}{\overline{S_{\vee}}}$
3.	1236	of type	$3+1$	
4.	1237	$\overline{S_{\wedge\vee}}$	and	$\underset{\wedge\vee}{\overline{S_{\wedge}}}$
5.	1256	$\underset{\wedge}{\overset{\wedge}{\overline{S_{\vee}}}}$		
6.	1257	of type	$3+1$	
7.	1258	$\overline{S_{\wedge}}$	and	$\overline{S_{\vee}}$
8.	1267	$\underset{\vee}{\overset{\wedge}{\overline{S'_{\vee}}}}$	and	$\underset{\wedge}{\overset{\vee}{\overline{S'_{\wedge}}}}$
9.	1268	of type	$3+1$	
10.	1278	$\overline{S_{\wedge}}$		
11.	1357	$\underset{\wedge\vee}{\overset{\vee}{\overline{S_{\wedge\vee}}}}$		
12.	1368	of type	$3+1$	

Table 1

From these twelve, four are of type 3+1 (having three absorption with the same operation outside the brackets). The other eight are representatives for the equivalence classes we found, in[2], as it is indicated in table 1. As it is mentioned in all the papers [1], [2], [3] there are examples that prove that these classes of noncommutative generalizations are distinct.

2 Closure operations

Let us consider an operator $a : \mathcal{P}(\omega_+) \to P(\omega_+)$ defined by:

$$a\xi = \{(X) \in \omega_+ \mid (X) \quad \text{is obeyed in any} \quad l \in \mathcal{L}_\xi\}.$$

We can easily verify that it is a closure operator on $\omega_+ = \omega \cup \{A_0, B_0\}$ (defined in the Introduction). In order to establish the exact value of $a\xi$, for every $\xi \subseteq \omega_+$, we will need other closure operators, which "approximate" up and down our operator a.

Our discussion from this section takes place in the general frame of a complete lattice with greatest element V, and it will be applied in the next sections for the complete lattice of systems of axioms $\mathcal{P}(\omega_+)$.

Following the Kalman ideas, we will consider a complete lattice L with greatest element V, G a group of lattice automorphisms $g : L \to L$ and $\mathcal{C}$ the set of all closure operations $c : L \to L$, which are "compatible with G", i.e. which are such that $xgc = xcg$, for all x and g in G. If we define an order relation " $\leq$ " by: $c \leq c'$ if and only if $xc \leq xc'$ for all x in L, it is easy to verify that $\mathcal{C}$ becomes a complete lattice. Let $\mathcal{Z}$ be the set of all subsets Z of L which are such that (i) $V \in Z$ (ii) if $x \subseteq Z$, then Inf $X \in Z$ and (iii) if $x \in Z$, then $xg \in Z$ for all g in G. The set $\mathcal{Z}$ becomes a complete lattice when for Z, Z' in $\mathcal{Z}$ we set $Z \subseteq Z'$ if and only if $Z \subseteq Z'$ (set theoretic inclusion). Also a dual isomorphism ψ of G onto $\mathcal{Z}$ may be defined by setting: $c\psi = \{x \mid x \in L \text{ and } x = xc\}$.

The inverse dual isomorphism ψ is given by:

$$x(Z\psi^{-1}) = Inf\{y \mid y \in Z \quad \text{and} \quad y \geq x\}$$

If c_0 is a partially defined unary operation on L i.e. a mapping of some subset L_0 of L into L, and if c in $\mathcal{C}$ is given by:

$$c = \text{Inf}\{b \mid b \in \mathcal{C} \quad \text{and} \quad xbc_0 \quad \text{for all} \quad x \in L_0\},$$

it can be easy verified that $c : L \to L$ is a closure operator. We will call c_0 a "G-support" of c. If Z_0 is any subset of L, and if Z in $\mathcal{Z}$ is given by:

$$Z = Inf\{W \mid W \in \mathcal{Z} \quad \text{and} \quad W \supseteq Z_0\},$$

we will call Z_0 a "G base" of the closure operation $Z\psi^{-1}$. If c is any closure operation on L and $x \in L$, we will say that x is "c"-closed if $x = xc$ and that x is "c-independent" if no y in L is such that $y < x$ and $yc = xc$.

Remark 1. If $c_1, c_2 : L \to L$ are two closure operation such that $c_1 \leq c_2$, then, for any $\xi \in L$,

a) ξ is c_1-dependent $\Rightarrow \xi$ is c_2-dependent

b) ξ is c_2-dependent $\Rightarrow \xi$ is c_1-independent.

Indeed, suppose there exist $y < \xi$ such that $yc_1 = \xi c_1$. Then we have $y < \xi \leq \xi c_1 = yc_1 \leq yc_2$, thus $\xi \leq yc_2$. Applying c_2 we have $\xi c_2 \leq yc_2$. The converse, inequality is obvious since $y < \xi$. Thus $\xi c_2 = yc_2$ and ξ is c_2 dependent

b) Results from a).

3 The compatibility of a with a group of lattice - automorphisms

On the family $\mathcal{L}$ of all algebraic systems $l = (L, \wedge, \vee)$ consisting of all the set L together with binary operations $\wedge$ and $\vee$, Kalman considered the transformations Π and ρ :

$$\pi : \mathcal{L} \to \mathcal{L},\ \rho : \mathcal{L} \to \mathcal{L},\ \forall\, l = (L, \wedge, \vee) \in \mathcal{L},\ l\Pi = (L, \wedge_\Pi, \vee_\Pi),\ l\rho = (L, \wedge_\rho, \vee_\rho)$$

where,

(9) $x \wedge_\pi y = y \vee x\ ,\ x \vee_\pi y = x \wedge y\ ,\ \forall\ x, y \in L$

(10) $x \wedge_\rho y = x \vee y\ ,\ x \vee_\rho y = x \wedge y\ ,\ \forall\ x, y \in L.$

It is easy to verify that $\rho\pi = \pi^3\rho$ and $\Pi^4 = \rho^2 = \varepsilon$ (the identity transformation). The transformation Π and ρ generate in the subgroup of all transformations on L, a subgroup Γ and all the elements of Γ can be written in at least one way in the form $\Pi^m\rho^n$, $m \in \{0, 1, 2, 3\}$, $n \in \{0, 1, 2\}$.

Kalman also considered two permutations p and q in the permutation group of the elements $1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, I, J, A_0, B_0$. We will consider in the same way two permutation p and q of the elements $1, 2, 3, 4, 5, 6, 7, 8,$ $A, B, C, D, I, J, A_0, B_0$:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 2 & 3 & 4 & 1 & 8 & 5 & 6 & 7 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$
$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & A & B & C & D & I & J & A_0 & B_0 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & B & A & D & C & J & I & B_0 & A_0 \end{pmatrix}$$

By the fact p correspond to the permutation Π we meant that p indicates the correspondence between the axioms fulfilled in an algebraic structure $l \in \mathcal{L}$ and the correspondent axioms that hold in $l\Pi$.

Analogously we determine q. It is easy to verify that, for all $l \in \mathcal{L}$ and $(X) \in \omega_+$; l obeys $(X) \Leftrightarrow l\Pi$ obeys $(X)p \Leftrightarrow l\rho$ obeys $(X)q$. Let P be the subgroups generated by p and q in the group of all permutations of the elements of ω_+. It is easily seen that the elements of p and q verify: $p^4 = q^2 = e$ and $qp = p^3q$.

If follows that the elements of p are precisely of the form $p^m q^n$, $m \in \{0,1,2,3\}$, $n \in \{0,1,2\}$ and that the mapping $\lambda : P \to \Gamma$, $\lambda(p^m \rho^n) = \Pi^m \rho^n$, $m \in \{0,1,2,3\}$, $n \in \{0,1,2\}$ is an homomorphism of P onto Γ (we will see in §4 that λ is in fact an isomorphism).

We give below the permutation subgroup generated by p and q

r	1	2	3	4	5	6	7	8	A	B	C	D	I	J	A_0	B_0
e	1	2	3	4	5	6	7	8	A	B	C	D	I	J	A_0	B_0
p	2	3	4	1	8	5	6	7	B	A	D	C	J	I	B_0	A_0
p^2	3	4	1	2	7	8	5	6	A	B	C	D	I	J	A_0	B_0
p^3	4	1	2	3	6	7	8	5	B	A	D	C	J	I	B_0	A_0
q	5	6	7	8	1	2	3	4	B	A	D	C	J	I	B_0	A_0
pq	6	7	8	5	4	1	2	3	A	B	C	D	I	J	A_0	B_0
p^2q	7	8	5	6	3	4	1	2	B	A	D	C	J	I	B_0	A_0
p^3q	8	5	6	7	2	3	4	1	A	B	C	D	I	J	A_0	B_0

Table 2.

Using the following:

- an axiom (X) is true in $l \Leftrightarrow$ the axiom $(X)p$ is true in $l\Pi = l(p\lambda)$
- an axiom (X) is true in $l \Leftrightarrow$ the axiom $(X)q$ is true in $l\rho = l(q\lambda)$
- λ is a homomorphism,

the following lemma hold:

Lemma 1. For all $l \in \mathcal{L}, (X)$ in ω_+ and $r \in P$, l obeys the law (X) if and only if $l(r\lambda)$ obeys the law $(X)r$.

If we choose a permutation $r \in P$, to each subset of axioms from ω_+ we can associate the corresponding subset of axioms, by r. Thus we have defined a transformation μ :

$$\xi(r\mu) = \{(Y) \mid \exists\, r \in P \quad \text{and} \quad \exists\, (X) \in \xi \quad \text{and such that} \quad (Y) = (X)r\}$$

having the domain P. $r\mu$ is a lattice automorphism of the Boolean algebra $P(\omega_+)$. If we denote by G the immage of μ, we have that μ is an isomorphism of P onto the group of automorphisms of $P(\omega_+)$.

Let's choose again a permutation $r \in P$. We notice that the operator a defined in §2 is, by it's definition, compatible with any transformation which associates to a $\xi \subseteq \omega_+$ the resulting subset $(\xi)r$ (immage of the set ξ by

r), namely $[(\xi)r]a = (\xi a)r$. On the other side, from definition of μ we have $\xi(r\mu) = (\xi)r$. Thus,

$$[\xi(r\mu)]a = [(\xi)r]a = (\xi a)r = (\xi a)(r\mu),\ \forall\ r\mu \in G,$$

namely we have:

Lemma 2. The closure operation a is compatible with G.

In the final of this paragraph we will define on ω_+ the relation: if $\xi, \eta \subseteq \omega_+$ are such that $\eta = \xi(r\mu)$ for some $r \in P$. we will call ξ and η "congruent" subsets of ω_+ and it follows from Lemma 2 that, if a subset ξ of ω_+ is a closed, [a-independent] then every η congruent to ξ is a-closed [a-independent].

4 Main result

If a partially defined unary operation c_0 on a given set has domain $\{\xi_1, \xi_2, \ldots, \xi_n\}$ and we have $\xi_i c_0 = \eta_i$, $i = \overline{1, n}$, we will say that c_0 has "defining relations" $\xi_1 \to \eta_1, \xi_2 \to \eta_2, \ldots, \xi_n \to \eta_n$. If the distinct elements of a nonempty subset ξ of ω are $(X_1), \ldots, (X_n)$, we will write $\xi = [X_1 X_2 \ldots X_n]$. Let a_0 be the partially defined operation on the subset of ω which has defined relations

$$\begin{aligned}&[A] \to [A_0], [C] \to [A_0], [12] \to [J], [15] \to [J], [1C] \to [8], [1D] \to [6],\\ &[17] \to [I], [123] \to [8], [127A_0] \to [8], [1267B_0] \to [D] \quad \text{and} \quad [1368BJ] \to [D],\end{aligned}$$

and let a_1 be the closure operation on the subsets of ω which has G-support a_0. The following lemma hold:

Lemma 3. $a_1 \leq a$.

Proof. It is sufficient to prove $\xi a \supseteq \xi a_0$ for each ξ in the domain of a_0.

In the presence of associativity (A) or commutativity (C), the axiom A_0: $(x \wedge y) \wedge x = x \wedge (y \wedge x)$ is obviously fulfilled. The following six implications and the last one are true by Lemma 3 from [1]. We must prove $[127A_0] \to [8]$ and $[1267B_0] \to [D]$.

First we must prove using (1), (2), (7) and (A_0) that (8): $(x \vee y) \wedge x = x$ is fulfilled. From [12] results [J] and from [$1J$] results (I). In any $l = (L, \wedge, \vee)$, from $\mathcal{L}[127A_0]$, for any $x, y \in L$, using A_0 we have first:

$$[x \wedge (x \vee y)] \wedge x = x \wedge [(x \vee y) \wedge x]$$

The member from left is equal to x by 1. After that we apply $\vee[(x \vee y) \wedge x]$ and thus:

$$x \vee [(x \vee y) \wedge x] = (x \vee y) \wedge x.$$

But by (2) the member from left is equal to x, and thus we obtain that (8) is true in L.

We will prove $[1267B_0]$ implies $[D]$.

$$x \vee y \overset{7}{=} [y \wedge (x \vee y)] \vee (x \vee y) \overset{6}{=} y \vee (x \vee y) \overset{B_0}{=} (y \vee x) \vee y.$$

Analogously $y \vee x = (x \vee y) \vee x$.

Using these, we have:

$$x \vee y = (y \vee x) \vee y = (y \vee x) \vee [y \wedge (y \vee x)] \overset{2}{=} y \vee x.$$

$\square$

Remark 2. The defining relations $[1267B] \to [D]$ and $[127A] \to [8]$ from the study of ω, in [1], were replaced, after Kalman's idea with $[1267B_0] \to [D]$ and $[127A_0] \to [B]$. Thus the associativity appears just in two of the defining relations of a_0.

Remark 3. We remark that the value of the operator a_1 can be calculated. We consider the defining relations of a_0 and their permutation obtained by table 2. There are 66 distinct relations, the set of which will be denoted by S. We consider then the operator $c : P(\omega_+) \to P(\omega_+)$ which acts as follows on a given $\varepsilon \subseteq \omega_+$: adds the conclusion of each from the 66 relations, if the respective hypothesis is found in ξ, replacing after that each time ξ with the result system. It's obvious that, there exits a natural number which depends of the given $\xi, n(\xi)$ such that $c^{n(\xi)}(\xi) = c^{n(\xi)+1}(\xi)$. If we consider $n = \sup\limits_{\xi \in P(\omega_+)} n(\xi)$, then , for any $\xi \subseteq \omega_+$

(11) $\xi a_1 = \xi c^n$

since c^n is a closure operator, verifies $c^n \geq a_0$ on the domain of a_0 and it is the least with these properties.

We will consider now a few algebraic structures $(L, \wedge, \vee)$ which will be counterexamples for certain implications between the subsets of ω_+. Kalman indicated in [1] the following examples (the sequences mean the rows of the corresponding multiplication tables for $\wedge$ and $\vee$):

$$L_1 = \{0,1,2,3,4\},\ 00000\ 01111\ 01202\ 01033\ 01234\ 01234\ 11234\ 22244\ 33434\ 44444$$
$$L_2 = \{0,1,2,3,4\},\ 00000\ 01111\ 01202\ 01133\ 01234\ 01234\ 11234\ 22244\ 33434\ 44444$$
$$L_3 = \{0,1,2\}, 000\ 011\ 012\ 022\ 112\ 222$$
$$L_4 = \{0,1\}, 01\ 01\ 00\ 11$$
$$L_5 = \{0,1,2\},\ 000\ 012\ 012\ 022\ 111\ 222$$
$$L_6 = \{0,1,2\},\ 010\ 011\ 012\ 022\ 212\ 222$$
$$L_7 = \{0,1\},\ 01\ 01\ 00\ 11$$
$$L_8 = \{0,1\},\ 01\ 10\ 00\ 00$$
$$L_9 = \{0,1\},\ 01\ 11\ 00\ 00$$
$$L_{10} = \{0,1,2\},\ 000\ 011\ 012\ 022\ 212\ 222$$
$$L_{11} = \{0,1,2\},\ 000\ 011\ 012\ 012\ 222\ 222$$
$$L_{12} = \{0,1,2\},\ 000\ 001\ 002\ 012\ 222\ 222$$
$$L_{13} = \{0,1,2,3\},\ 0000\ 0111\ 0122\ 0123\ 0233\ 3133\ 3323\ 3333$$

Remark 4. a) L_2 doesn't verify A_0. Indeed for $x = 2$ and $y = 3$, $(x \wedge y) \wedge x = x \wedge (y \wedge x) \Leftrightarrow 0 = 1$.

b) L_3 doesn't verify B_0. Indeed, for $x = 1$, $y = 0$, we have $(x \vee y) \vee x = x \vee (y \vee x) \Leftrightarrow 1 = 2$.

c) L_{14} satisfies $[1368ACIJA_0B_0]$ and doesn't satisfies the rest of the axioms ω_+. Indeed $(L_{14}, \wedge)$ is the restrictive semigroup of the chain $0 \leq 1 \leq 2 \leq 3$ and it verifies $[ACIA_0]$. The rest it is easy to verify.

Let z_0 be the following family of subsets of ω_+.

$$\aleph_1 = [12345678BCDIJA_0B_0]$$
$$\aleph_2 = [12345678BDIJB_0]$$
$$\aleph_3 = [123568ACIJA_0]$$
$$\aleph_4 = [123578ABIJA_0B_0]$$
$$\aleph_5 = [12358ABIJA_0B_0]$$
$$\aleph_6 = [123678ABDIJA_0B_0]$$
$$\aleph_7 = [1258ABCIJA_0B_0]$$

$$\aleph_8 = [1368ABCDA_0B_0]$$
$$\aleph_9 = [1368ABCDIA_0B_0]$$
$$\aleph_{10} = [1368ABCDIJA_0B_0]$$
$$\aleph_{11} = [1368ABCIA_0B_0]$$
$$\aleph_{12} = [17ABA_0B_0]$$
$$\aleph_{13} = [1368ACIJA_0B_0]$$

and let a_2 be the closure operation on the subsets of which has G-base Z_0.

Lemma 4. $a \leq a_2$.

Proof. Since Z_0 is a G base of a_2, we have:

$$a_2\psi = Z = \mathrm{Inf}\{W \mid W \in 3\mathcal{Z} \quad \text{and} \quad W \supseteq Z_0\}$$

and in the same time we know from the definition of ψ that Z is the set of all elements from $P(\omega_+)$ which are a_2-closed, thus, for any $\xi \subseteq \omega_+$,

(12) $\xi a_2 = \cap\{y \in Z \mid y \supseteq \xi\}$

We have to prove $\xi a \leq \cap\{y \in Z \mid y \supseteq \xi\},\ \forall\, \xi \subseteq \omega_+$.

We notice from (12) and from definition of $\mathcal{Z}$ that

$$Z = Z_0 \cup \bigcup_{g\in G} g(Z_0) \cap \{\omega_+\} \cup F,$$

where F denotes the sets of the form inf X, where $X \subseteq Z_0 \cup \bigcup\limits_{g\in G} g(Z_0) \cup \{\omega_+\}$.

Thus

(13) $\xi a_2 = \cap\{y \in Z_0 \cup \bigcup_{g\in G} g(Z_0) \cup \{\omega_+\} \mid y \supseteq \xi\}$

If we prove that all the elements considered in the last expression are a-closed, then the lemma is true, since each such y will satisfy

$$y = ya \supseteq \xi a.$$

The element ω_+ is obviously a-closed, and if the elements of Z_0 are a-closed, then also the elements of $g(Z_0)$ are a-closed, since a is compatible with any $g \in G$. The fact that the elements of Z_0 are a-closed results from the fact that the algebraic structures L_i, $i = \overline{1,13}$ presented in this paragraph verify $[\aleph_i]$ but don't verify $\omega_+ \setminus \aleph_i$. □

Remark 5. We remark that the values of the operator a_2 can be calculated. If we consider the systems $\aleph_1, \aleph_2, \ldots, \aleph_{13}$ and the permutated systems obtained from them by table 2, we obtain 104 systems $\aleph_1, \aleph_2, \ldots, \aleph_{104}$. By definition of G (see §3) we have then:

(14) $\xi a_2 \cap \{y \in \{\aleph_1, \aleph_2, \ldots, \aleph_{104}\} \mid y \supseteq \xi\}$

We now state the theorems which are the main results of this paper.

Theorem 1. The operation a_0 is a G-support of a and the family Z_0 is a G base of a.

Proof. To prove theorem 1 it will be sufficient to show that $a_1 = a_2$. Then, by Lemma 3 and 4, we will have $a_1 = a = a_2$, completing the proof.

Let us consider $\theta \subseteq \omega_+$, arbitrary systems of axioms.

The computer programm calculates θa_1 and θa_2 by (11) and (13) and compares the results. For any $\theta \subseteq \omega_+$ we have $\theta a_1 = \theta a_2$ and thus $a_1 = a = a_2$. □

Let's denote by $\overline{\theta}$ the common value of θa_1 and θa_2.

Theorem 2. A subset of ω_+ is a-independent if and only if it is congruent to one of the subsets θ listed in column θ of table 3. The entry in the row of a certain θ and column $\overline{\theta}$ of table 3 is θa.

Note It may easily be checked that each entry in column θ of table 3 is the lexicographically first element of it's congruence class. Thus, no two subsets in column 0 are congruent to each other.

Proof. Since for each $\theta \subseteq \omega_+$ we know the exact value of θa as we explained in the proof of theorem 1 means that we can establish, by a procedure with an element $\theta \subseteq \omega_+$ is a-independent. Calculating the value of a for the subsystems of θ. □

5 About the programm

The programm has got a few functions. The must important are:

- a function "minim" which receive a sequence representing a system of axioms, calculates the elements from the same congruence class using table 2, and returns the least sequence in lexicographical order.
- a function "aplică t" which calculates a_1 for a given sequence θ, using (11).
- a function "aplicaă 2" which calculates a_2 for a given sequence θ, using (13). The matrix $\aleph$ having the rows $\aleph_1, \ldots, \aleph_{104}$ is taken from the main programm and it is generated using table 2 by another function
- a function "verifindep" which verifies if a given system is a-independent.

The main programm generates the subsets of ω_+ in lexicographical order. When each system θ is formed, it is "minimized", by the function "minim". After that the programm verifies the independency of the resulted system, called "min." We can renounce first at verifying the independency and we want to see first that $a_1 = a_2$ as we explained in the proof of the Theorem 1. In this way we follow the logic order of the ideas. In the case when we verify the independency,

when the system θ is independent it is put in a list, introducing it where it is it's place in lexicographical order. After the last element $\xi \subset \omega_+$ has been generated (this is $[B_0]$) and it is verified it's independency, the programm starts to print the list of a-independent systems. After printing $\theta \subseteq \omega_+$ from the list, it calculates θa_1 by the function "aplicat" and θ_2 by the function "aplica 2", verifies if θa_1 and θa_2 coincide and if not, it gives us a message and stops running. No such message has been received and thus $\theta a_1 = \theta a_2$.

In the case of the coincidence, it prints θa_1 and goes further to the next θ from the list.

The table 3, of the results, contains 599 rows for all the 599 independent systems found in ω_+.

For a simple writing of the results, the programm prints the letters "k" and "l" instead of notations "(A_0)" and "(B_0)". The axioms $(A), (B), (C), (D), (I), (J)$ are denoted in the list of the results with small letters.

Let's interpret the results for two systems of axioms.

The system [1234]:

-is independent since appears in the first coloumn of the tabel 3.

- implies the axioms $1, 2, 3, 4, 5, 6, 7, 8, I, J$ and no other axioms from ω_+.

- together with the axiom (A), implies $[12345678ACIJA_0]$.

-together with the axiom (C), implies $[12345678CIJA_0]$.

-together with the axiom (A_0) (written in the tabel as k), implies $[12345678CIJA_0]$ namely the same system of axioms as in the case if we had added (C), and the same system of absorption, commutativity, and idempotence axioms as in the case if we had added (A).

The system [1368]:

-is independent since appears in the first coloumn of the tabel 3.

-implies the axioms $1, 3, 6, 8$ and no other axioms from ω_+ .

- together with the axiom (A) it implies $[1368AA_0]$.

- together with the axiom (C) it doesn't form an independent subsystem and we must find a subsystem of $[1368C]$ which is independent and implies these axioms. We find $[13C]$, which implies $[1368CA_0]$ and no other axioms beside these.

-together with the axiom (A_0), it forms an independent system, and it implies just the same system $[1368A_0]$. Thus, from the point of view of absorption, commutativity and idempotence axioms that result, we have the same result if we add (A) or (A_0), but we don't have the same result if we add (C) or (A_0).

tetha	tetha-bar
1	1
12	12ij
123	1238ij
1234	12345678ij
1234a	12345678acijk
1234ab	12345678abcdijkl
1234al	12345678acdijkl
1234c	12345678cijk
1234k	12345678cijk
1234kl	12345678cdijkl
1235	12358ij
12356	123568ij
123567	12345678ij
12356a	123568acijk
12356ab	12345678abcdijkl
12356al	12345678acdijkl
12356b	12345678bdijl
12356bk	12345678bcdijkl
12356k	123568cijk
12356kl	12345678cdijkl
12356l	12345678dijl
12357	123578ij
12357a	123578aijk
12357ab	123578abijkl
12357al	123578aijkl
12357b	123578bijl
12357bk	123578bijkl
12357k	123578ijk
12357kl	123578ijkl
12357l	123578ijl
1235a	12358aijk
1235ab	12358abijkl
1235al	12358aijkl
1235b	12358bijl
1235bd	12345678bdijl
1235bk	12358bijkl
1235d	12345678dijl
1235k	12358ijk
1235kl	12358ijkl
1235l	12358ijl
1236	12368ij
12367	123678ij
1236a	12368aijk
1236ab	123678abdijkl

1236al	123678adijkl
1236b	123678bdijl
1236bk	123678bdijkl
1236k	12368ijk
1236kl	123678dijkl
1236l	123678dijl
1237	12378ij
1237a	12378aijk
1237ab	12378abijkl
1237ac	12345678acijk
1237al	12378aijkl
1237b	12378bijl
1237bk	12378bijkl
1237c	12345678cijk
1237k	12378ijk
1237kl	12378ijkl
1237l	12378ijl
123a	1238aijk
123ab	1238abijkl
123abc	12345678abcdijkl
123ac	123568acijk
123acl	12345678acdijkl
123al	1238aijkl
123b	1238bijl
123bc	12345678bcdijkl
123bd	123678bdijl
123bk	1238bijkl
123c	123568cijk
123cl	12345678cdijkl
123d	123678dijl
123k	1238ijk
tetha	tetha-bar
123kl	1238ijkl
123l	1238ijl
125	125ij
1256	1256ij
1256a	1256aijk
1256ab	1256abijkl
1256al	1256aijkl
1256k	1256ijk
1256kl	1256ijkl
1257	1257ij
1257a	12578aijk
1257ab	12578abijkl
1257al	12578aijkl
1257b	1257bijl
1257bk	12578bijkl
1257k	12578ijk
1257kl	12578ijkl

1257l	1257ijl
1258	1258ij
1258a	1258aijk
1258ab	1258abijkl
1258al	1258aijkl
1258b	1258bijl
1258bd	12345678bdijl
1258bk	1258bijkl
1258d	12345678dijl
1258k	1258ijk
1258kl	1258ijkl
1258l	1258ijl
125a	125aijk
125ab	125abijkl
125abd	12345678abcdijkl
125ad	12345678acdijkl
125al	125aijkl
125b	125bijl
125bd	124567bdijl
125bdk	12345678bcdijkl
125bk	125bijkl
125d	124567dijl
125dk	12345678cdijkl
125k	125ijk
125kl	125ijkl
125l	125ijl
126	126ij
1267	1267ij
1267a	123678aijk
1267ab	123678abdijkl
1267ac	12345678acijk
1267al	123678adijkl
1267b	1267bdijl
1267bk	123678bdijkl
1267c	12345678cijk
1267k	123678ijk
1267kl	123678dijkl
1267l	1267dijl
1268	1268ij
1268a	1268aijk
1268ab	1268abijkl
1268al	1268aijkl
1268b	1268bijl
1268bk	1268bijkl
1268k	1268ijk
1268kl	1268ijkl
1268l	1268ijl
126a	126aijk
126ab	126abijkl

126abc	12345678abcdijkl
126ac	123568acijk
126acl	12345678acdijkl
126al	126aijkl
126b	126bijl
126bc	12345678bcdijkl
126bk	126bijkl
126c	123568cijk
126cl	12345678cdijkl
teha	tetha-bar
126k	126ijk
126kl	126ijkl
126l	126ijl
127	127ij
1278	1278ij
127a	1278aijk
127ab	1278abijkl
127ac	124578acijk
127al	1278aijkl
127b	127bijl
127bk	1278bijkl
127c	124578cijk
127k	1278ijk
127kl	1278ijkl
127l	127ijl
128	128ij
128a	128aijk
128ab	128abijkl
128al	128aijkl
128b	128bijl
128bd	123678bdijl
128bk	128bijkl
128d	123678dijl
128k	128ijk
128kl	128ijkl
128l	128ijl
12a	12aijk
12ab	12abijkl
12abc	1258abcijkl
12abcd	12345678abcdijkl
12abd	123678abdijkl
12ac	1258acijk
12acd	12345678acdijkl
12acl	1258acijkl
12ad	123678adijkl
12al	12aijkl
12b	12bijl
12bc	1258bcijkl
12bcd	12345678bcdijkl

12bd	1267bdijl
12bdk	123678bdijkl
12bk	12bijkl
12c	1258cijk
12cd	12345678cdijkl
12cl	1258cijkl
12d	1267dijl
12dk	123678dijkl
12k	12ijk
12kl	12ijkl
12l	12ijl
13	13
135	135ij
1357	1357ij
1357a	1357aijk
1357ab	1357abijkl
1357ac	12345678acijk
1357al	1357aijkl
1357c	12345678cijk
1357k	1357ijk
1357kl	1357ijkl
135a	135aijk
135ab	135abijkl
135abc	12345678abcdijkl
135ac	123568acijk
135acl	12345678acdijkl
135al	135aijkl
135b	135bijl
135bc	12345678bcdijkl
135bd	134568bdijl
135bk	135bijkl
135c	123568cijk
135cl	12345678cdijkl
135d	134568dijl
135k	135ijk
135kl	135ijkl
tetha	tetha-bar
135l	135ijl
136	136
1368	1368
1368a	1368ak
1368ab	1368abkl
1368abi	1368abikl
1368abj	1368abdijkl
1368ai	1368aik
1368ail	1368aikl
1368aj	1368aijk
1368ajl	1368aijkl
1368al	1368akl

1368b	1368bl
1368bi	1368bil
1368bik	1368bikl
1368bj	1368bdijl
1368bjk	1368bdijkl
1368bk	1368bkl
1368i	1368i
1368ik	1368ik
1368ikl	1368ikl
1368il	1368il
1368j	1368ij
1368jk	1368ijk
1368jkl	1368ijkl
1368jl	1368ijl
1368k	1368k
1368kl	1368kl
1368l	1368l
136a	136ak
136ab	136abkl
136abi	136abikl
136abj	136abijkl
136ai	136aik
136ail	136aikl
136aj	136aijk
136ajl	136aijkl
136al	136akl
136b	136bl
136bi	136bil
136bik	136bikl
136bj	136bijl
136bjk	136bijkl
136bk	136bkl
136i	136i
136ik	136ik
136ikl	136ikl
136il	136il
136j	136ij
136jk	136ijk
136jkl	136ijkl
136jl	136ijl
136k	136k
136kl	136kl
136l	136l
13a	13ak
13ab	13abkl
13abc	1368abckl
13abci	1368abcikl
13abcj	1368abcdijkl
13abd	1368abdkl

13abdi	1368abdikl
13abdj	1368abdijkl
13abi	13abikl
13abj	13abijkl
13ac	1368ack
13aci	1368acik
13acil	1368acikl
13acj	1368acijk
13acjl	1368acijkl
13acl	1368ackl
13ad	1368adkl
13adi	1368adikl
13adj	1368adijkl
13ai	13aik
tetha	tetha-bar
13ail	13aikl
13aj	13aijk
13ajl	13aijkl
13al	13akl
13b	13bl
13bc	1368bckl
13bci	1368bcikl
13bcj	1368bcdijkl
13bd	1368bdl
13bdi	1368bdil
13bdik	1368bdikl
13bdj	1368bdijl
13bdjk	1368bdijkl
13bdk	1368bdkl
13bi	13bil
13bik	13bikl
13bj	13bijl
13bjk	13bijkl
13bk	13bkl
13c	1368ck
13ci	1368cik
13cil	1368cikl
13cj	1368cijk
13cjl	1368cijkl
13cl	1368ckl
13d	1368dl
13di	1368dil
13dik	1368dikl
13dj	1368dijl
13djk	1368dijkl
13dk	1368dkl
13i	13i
13ik	13ik
13ikl	13ikl

13il	13il
13j	13ij
13jk	13ijk
13jkl	13ijkl
13jl	13ijl
13k	13k
13kl	13kl
13l	13l
15	15ij
15a	15aijk
15ab	15abijkl
15abc	1258abcijkl
15abcd	12345678abcdijkl
15ac	1258acijk
15acd	12345678acdijkl
15acl	1258acijkl
15ad	1456adijkl
15al	15aijkl
15c	1258cijk
15cd	12345678cdijkl
15cl	1258cijkl
15k	15ijk
15kl	15ijkl
16	16
16a	16ak
16ab	16abkl
16abc	1368abckl
16abci	1368abcikl
16abcj	1368abcdijkl
16abi	16abikl
16abj	16abijkl
16ac	1368ack
16aci	1368acik
16acil	1368acikl
16acj	1368acijk
16acjl	1368acijkl
16acl	1368ackl
16ai	16aik
16ail	16aikl
16aj	16aijk
16ajl	16aijkl
tetha	tetha-bar
16al	16akl
16b	16bl
16bc	1368bckl
16bci	1368bcikl
16bcj	1368bcdijkl
16bi	16bil
16bik	16bikl

16bj	16bijl
16bjk	16bijkl
16bk	16bkl
16c	1368ck
16ci	1368cik
16cil	1368cikl
16cj	1368cijk
16cjl	1368cijkl
16cl	1368ckl
16i	16i
16ik	16ik
16ikl	16ikl
16il	16il
16j	16ij
16jk	16ijk
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16jl	16ijl
16k	16k
16kl	16kl
16l	16l
17	17
17a	17ak
17ab	17abkl
17abc	124578abcijkl
17abcd	12345678abcdijkl
17abi	17abijkl
17ac	1478acijk
17acd	12345678acdijkl
17acl	124578acijkl
17ad	123678adijkl
17ai	17aijk
17ail	17aijkl
17aj	17aijk
17ajl	17aijkl
17al	17akl
17c	1478cijk
17cd	12345678cdijkl
17cl	124578cijkl
17i	17ij
17ik	17ijk
17ikl	17ijkl
17il	17ijl
17k	17k
17kl	17kl
18	18
18a	18ak
18ab	18abkl
18abd	1368abdkl
18abdi	1368abdikl

18abdj	1368abdijkl
18abi	18abikl
18abj	18abijkl
18ad	1368adkl
18adi	1368adikl
18adj	1368adijkl
18ai	18aik
18ail	18aikl
18aj	18aijk
18ajl	18aijkl
18al	18akl
18b	18bl
18bd	1368bdl
18bdi	1368bdil
18bdik	1368bdikl
18bdj	1368bdijl
18bdjk	1368bdijkl
18bdk	1368bdkl
18bi	18bil
tetha	tetha-bar
18bik	18bikl
18bj	18bijl
18bjk	18bijkl
18bk	18bkl
18d	1368dl
18di	1368dil
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18dj	1368dijl
18djk	1368dijkl
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18ik	18ik
18ikl	18ikl
18il	18il
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1ab	1abkl
1abc	18abckl
1abcd	1368abcdkl
1abcdi	1368abcdikl
1abcdj	1368abcdijkl
1abci	18abcikl
1abcj	18abcijkl

1abd	16abdkl
1abdi	16abdikl
1abdj	16abdijkl
1abi	1abikl
1abj	1abijkl
1ac	18ack
1acd	1368acdkl
1acdi	1368acdikl
1acdj	1368acdijkl
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1acil	18acikl
1acj	18acijk
1acjl	18acijkl
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1bcd	1368bcdkl
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1bcj	18bcijkl
1bd	16bdl
1bdi	16bdil
1bdik	16bdikl
1bdj	16bdijl
1bdjk	16bdijkl
1bdk	16bdkl
1bi	1bil
1bik	1bikl
1bj	1bijl
1bjk	1bijkl
1bk	1bkl
1c	18ck
1cd	1368cdkl
1cdi	1368cdikl
1cdj	1368cdijkl
1ci	18cik
1cil	18cikl
tetha	tetha-bar
1cj	18cijk
1cjl	18cijkl

1cl	18ckl
1d	16dl
1di	16dil
1dik	16dikl
1dj	16dijl
1djk	16dijkl
1dk	16dkl
1i	1i
1ik	1ik
1ikl	1ikl
1il	1il
1j	1ij
1jk	1ijk
1jkl	1ijkl
1jl	1ijl
1k	1k
1kl	1kl
1l	1l
a	ak
ab	abkl
abc	abckl
abcd	abcdkl
abcdi	abcdikl
abcdij	abcdijkl
abci	abcikl
abcij	abcijkl
abcj	abcjkl
abi	abikl
abij	abijkl
ac	ack
acd	acdkl
acdi	acdikl
acdij	acdijkl
acdj	acdjkl
aci	acik
acij	acijk
acijl	acijkl
acil	acikl
acj	acjk
acjl	acjkl
acl	ackl
ad	adkl
adi	adikl
adij	adijkl
adj	adjkl
ai	aik
aij	aijk
aijl	aijkl
ail	aikl

aj	ajk
ajl	ajkl
al	akl
c	ck
cd	cdkl
cdi	cdikl
cdij	cdijkl
ci	cik
cij	cijk
cijl	cijkl
cil	cikl
cj	cjk
cjl	cjkl
cl	ckl
i	i
ij	ij
ijk	ijk
ijkl	ijkl
ik	ik
ikl	ikl
il	il
k	k
kl	kl
nr. lines	599.000000

Table 3

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